

A Nash-Moser-Hörmander implicit function theorem with applications to control and Cauchy problems for PDEs

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Abstract. We prove an abstract Nash-Moser implicit function theorem which, when applied to control and Cauchy problems for PDEs in Sobolev class, is sharp in terms of the loss of regularity of the solution of the problem with respect to the data. The proof is a combination of: (i) the iteration scheme by Hörmander (ARMA 1976), based on telescoping series, and very close to the original one by Nash; (ii) a suitable way of splitting series in scales of Banach spaces, inspired by a simple, clever trick used in paradifferential calculus (for example, by Métivier). As an example of application, we apply our theorem to a control and a Cauchy problem for quasi-linear perturbations of KdV equations, improving the regularity of a previous result. With respect to other approaches to control and Cauchy problems, the application of our theorem requires lighter assumptions to be verified. *MSC2010:* 47J07, 35Q53, 35Q93.

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1 Introduction

In this paper we prove an abstract Nash-Moser implicit function theorem (Theorem 2.1) which, when applied to control and Cauchy problems for evolution PDEs in Sobolev class, is sharp in terms of the loss of regularity of the solution of the problem with respect to the data.

In terms of such a loss, the sharpest Nash-Moser theorem in literature seems to be the one by Hörmander (Theorem 2.2.2 in Section 2.2 of [19], and main Theorem in [20]). Hörmander’s theorem is sharp when applied to PDEs in Hölder spaces (with non-integer exponent), but it is *almost* sharp in Sobolev class: if the approximate right inverse of the linearized operator loses γ derivatives, and the data of the problem belong to H^s , then the application of Hörmander’s theorem gives solutions of regularity $H^{s-\gamma-\varepsilon}$ for all $\varepsilon > 0$, whereas one expects to find $H^{s-\gamma}$ (and in many cases, with other techniques, in fact one can prove such a sharp regularity). Our Theorem 2.1 applies to Sobolev spaces with sharp loss, and thus it extends Hörmander’s result to Sobolev spaces.

As it is well-known, the Nash-Moser approach is natural to use in situations where a loss of regularity prevents the application of other, more standard iteration schemes (contractions, implicit function theorem, schemes based on Duhamel principle, etc.). Typical situations where such a loss is unavoidable are related, for example, to the presence of the so-called “small divisors”. In addition to that, sometimes it could be convenient to use a Nash-Moser iteration even if other techniques are also available. In general, the advantages of the Nash-Moser method for nonlinear PDEs (especially quasi-linear ones) with respect to other approaches are essentially these: the required estimates on the solution of the linearized problem allow some loss of regularity, also with respect to the coefficients;

the continuity of the solution of the linearized problem with respect to the linearization point is not required for the existence proof; linearizing does not introduce nonlocal terms (whereas, for example, in some other schemes parilinearizing does); the nonlinear scheme is “packaged” in the theorem and ready-to-use, and its application to a PDE problem reduces to verify its assumptions, which mainly consists of a careful analysis of the linearized operator.

Without claiming to be complete, Nash-Moser schemes in Cauchy problems for nonlinear PDEs (especially with derivatives in the nonlinearity) have been used, for example, by Klainerman [22, 23] and, more recently, Lindblad [25], Alvarez-Samaniego and Lannes [5, 24], Alexandre, Wang, Xu and Yang [4] (see also Mouhot-Villani [29]) and, in control problems, by Beauchard, Coron, Alabau-Boussouira, Olive [9, 11, 10, 1] (a discussion about Nash-Moser method in the context of controllability of PDEs can be found in [14], section 4.2.2).

The Nash-Moser theorem was first introduced by Nash [30], then many refinements, improvements and new versions were developed afterwards: without demanding completeness we mention, for example, the results by Moser [27], Zehnder [32], Hamilton [18], Gromov [17], Hörmander [19, 20, 21], Alinhac and Gérard [3], and, more recently, Berti, Bolle, Corsi and Procesi [12, 13], Texier and Zumbrun [31], Ekeland and Séré [15, 16].

The iteration scheme by Hörmander [19] (based on telescoping series, and very close to the original scheme by Nash) is the one used for Cauchy problems by Klainerman [22, 23] and by Lindblad [25]. Hörmander’s theorem in [19] is formulated in the setting of Hölder spaces, and it also holds for other families of Banach spaces satisfying the same set of basic properties. Instead, Sobolev spaces do not satisfy that set of properties (see Remark 2.6). The same point is expressed, in other words, in [20, 21]. The theorems in [20] and [21] are formulated as abstract results, with sharp loss of regularity, in the class of *weak* Banach spaces E'_a , which Hörmander defines, using smoothing operators, starting from some given scale of Banach spaces E_a , $a \geq 0$. A key point is that if E_a is a Hölder space (with exponent $a \notin \mathbb{N}$), then it coincides with its weak counterpart E'_a , with equivalent norms (this is stated explicitly in [20], and proved implicitly in [19]). On the contrary, if E_a is a Sobolev space, then E'_a is a strictly larger set, with a strictly weaker norm (see Remark 2.5). What is true in Sobolev class is that $E_a \subset E'_a \subset E_b$ for all $b < a$, with continuous inclusions. This is the reason why the application of Hörmander’s theorems in Sobolev class produces a further, unavoidable, arbitrarily small loss. This further loss is not present if the theorems of [20, 21] are applied in the weak spaces E'_a , but these E'_a are not the usual Sobolev spaces (see also Remark 1.2 in [7]).

In Theorem 2.1 we overcome this issue by modifying the iteration scheme of [19], inspired by a trick commonly used in paradifferential calculus (see for example the proof of Proposition 4.1.13 on page 53 of [26]).

Theorem 2.1 is stated in Section 2, and it is followed by several comments and technical remarks. Its proof is contained in Section 3. An application of the theorem is given in Section 4, where we remove the loss of regularity from the results in [7] about control and Cauchy problems for quasi-linear perturbations of the Korteweg-de Vries equation in Sobolev class (Theorems 4.1 and 4.2). Possible applications to other PDEs are also mentioned (Remark 4.4).

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2 A Nash-Moser-Hörmander theorem

Let $(E_a)_{a \geq 0}$ be a decreasing family of Banach spaces with continuous injections $E_b \hookrightarrow E_a$,

$$\|u\|_a \leq \|u\|_b \quad \text{for } a \leq b. \quad (2.1)$$

Set $E_\infty = \cap_{a \geq 0} E_a$ with the weakest topology making the injections $E_\infty \hookrightarrow E_a$ continuous. Assume that $S_j : E_0 \rightarrow E_\infty$ for $j = 0, 1, \dots$ are linear operators such that, with constants C bounded when a and b are bounded, and independent of j ,

$$\|S_j u\|_a \leq C \|u\|_a \quad \text{for all } a; \quad (2.2)$$

$$\|S_j u\|_b \leq C 2^{j(b-a)} \|S_j u\|_a \quad \text{if } a < b; \quad (2.3)$$

$$\|u - S_j u\|_b \leq C 2^{-j(a-b)} \|u - S_j u\|_a \quad \text{if } a > b; \quad (2.4)$$

$$\|(S_{j+1} - S_j)u\|_b \leq C 2^{j(b-a)} \|(S_{j+1} - S_j)u\|_a \quad \text{for all } a, b. \quad (2.5)$$

From (2.3)-(2.4) one can obtain the logarithmic convexity of the norms

$$\|u\|_{\lambda a + (1-\lambda)b} \leq C \|u\|_a^\lambda \|u\|_b^{1-\lambda} \quad \text{if } 0 < \lambda < 1. \quad (2.6)$$

Set

$$R_0 u := S_1 u, \quad R_j u := (S_{j+1} - S_j)u, \quad j \geq 1. \quad (2.7)$$

Thus

$$\|R_j u\|_b \leq C 2^{j(b-a)} \|R_j u\|_a \quad \text{for all } a, b. \quad (2.8)$$

Bound (2.8) for $j \geq 1$ is (2.5), while, for $j = 0$, it follows from (2.1) and (2.3).

We also assume that

$$\|u\|_a^2 \leq C \sum_{j=0}^{\infty} \|R_j u\|_a^2 \quad \forall a \geq 0, \quad (2.9)$$

with C bounded for a bounded. This is a sort of “orthogonality property” of the smoothing operators.

Now let us suppose that we have another family F_a of decreasing Banach spaces with smoothing operators having the same properties as above. We use the same notation also for the smoothing operators.

Theorem 2.1. *Let $a_1, a_2, \alpha, \beta, a_0, \mu$ be real numbers with*

$$0 \leq a_0 \leq \mu \leq a_1, \quad a_1 + \frac{\beta}{2} < \alpha < a_1 + \beta, \quad 2\alpha < a_1 + a_2. \quad (2.10)$$

Let V be a convex neighborhood of 0 in E_μ . Let Φ be a map from V to F_0 such that $\Phi : V \cap E_{a+\mu} \rightarrow F_a$ is of class C^2 for all $a \in [0, a_2 - \mu]$, with

$$\begin{aligned} \|\Phi''(u)[v, w]\|_a &\leq M_1(a) (\|v\|_{a+\mu} \|w\|_{a_0} + \|v\|_{a_0} \|w\|_{a+\mu}) \\ &\quad + \{M_2(a) \|u\|_{a+\mu} + M_3(a)\} \|v\|_{a_0} \|w\|_{a_0} \end{aligned} \quad (2.11)$$

for all $u \in V \cap E_{a+\mu}$, $v, w \in E_{a+\mu}$, where $M_i : [0, a_2 - \mu] \rightarrow \mathbb{R}$, $i = 1, 2, 3$, are positive, increasing functions. Assume that $\Phi'(v)$, for $v \in E_\infty \cap V$ belonging to some ball $\|v\|_{a_1} \leq \delta_1$, has a right inverse $\Psi(v)$ mapping F_∞ to E_{a_2} , and that

$$\|\Psi(v)g\|_a \leq L_4(a) \|g\|_{a+\beta-\alpha} + \{L_5(a) \|v\|_{a+\beta} + L_6(a)\} \|g\|_0 \quad \forall a \in [a_1, a_2], \quad (2.12)$$

where $L_i : [a_1, a_2] \rightarrow \mathbb{R}$, $i = 4, 5, 6$, are positive, increasing functions.

Then for all $A > 0$ there exists $\delta > 0$ such that, for every $g \in F_\beta$ satisfying

$$\sum_{j=0}^{\infty} \|R_j g\|_\beta^2 \leq A^2 \|g\|_\beta^2, \quad \|g\|_\beta \leq \delta, \quad (2.13)$$

there exists $u \in E_\alpha$ solving $\Phi(u) = \Phi(0) + g$. The solution u satisfies

$$\|u\|_\alpha \leq CL_{456}(a_2)(1 + A)\|g\|_\beta,$$

where $L_{456} = L_4 + L_5 + L_6$ and C is a constant depending on a_1, a_2, α, β . The constant δ is

$$\delta = 1/B, \quad B = C' L_{456}(a_2) \max \{1/\delta_1, 1 + A, (1 + A)L_{456}(a_2)M_{123}(a_2 - \mu)\}$$

where $M_{123} = M_1 + M_2 + M_3$ and C' is a constant depending on a_1, a_2, α, β .

Moreover, let $c > 0$ and assume that (2.11) holds for all $a \in [0, a_2 + c - \mu]$, $\Psi(v)$ maps F_∞ to E_{a_2+c} , and (2.12) holds for all $a \in [a_1, a_2 + c]$. If g satisfies (2.13) and, in addition, $g \in F_{\beta+c}$ with

$$\sum_{j=0}^{\infty} \|R_j g\|_{\beta+c}^2 \leq A_c^2 \|g\|_{\beta+c}^2 \quad (2.14)$$

for some A_c , then the solution u belongs to $E_{\alpha+c}$, with

$$\|u\|_{\alpha+c} \leq C_c \{ \mathcal{G}_1(1 + A)\|g\|_\beta + \mathcal{G}_2(1 + A_c)\|g\|_{\beta+c} \} \quad (2.15)$$

where

$$\mathcal{G}_1 := \tilde{L}_6 + \tilde{L}_{45}(\tilde{L}_6 \tilde{M}_{12} + L_{456}(a_2) \tilde{M}_3) \sum_{j=0}^{N-2} z^j, \quad \mathcal{G}_2 := \tilde{L}_{45} \sum_{j=0}^{N-1} z^j, \quad (2.16)$$

$$z := L_{456}(a_1)M_{123}(0) + \tilde{L}_{45} \tilde{M}_{12}, \quad (2.17)$$

$\tilde{L}_{45} := \tilde{L}_4 + \tilde{L}_5$, $\tilde{L}_i := L_i(a_2 + c)$, $i = 4, 5, 6$; $\tilde{M}_{12} := \tilde{M}_1 + \tilde{M}_2$, $\tilde{M}_i := M_i(a_2 + c - \mu)$, $i = 1, 2, 3$; N is a positive integer depending on c, a_1, α, β ; and C_c depends on $a_1, a_2, \alpha, \beta, c$.

2.1 Comments

Remark 2.2. We underline that, in the higher regularity case $g \in F_{\beta+c}$, the smallness assumption $\|g\|_\beta \leq \delta$ is only required in “low” norm in Theorem 2.1 (and δ is independent of c). \square

Remark 2.3. If the first inequality in (2.13) does not hold, then one can apply Theorem 2.2.2 in [19] or Theorem 7.1 in [7], obtaining the same type of result with a small additional loss of regularity. The same if (2.9) does not hold. \square

Remark 2.4. With respect to the implicit function theorems in [19, 20, 7], in Theorem 2.1 we slightly modify the form of the tame estimates concerning Φ'' and Ψ , allowing the presence of extra terms, corresponding to $M_3(a)$ in (2.11) and $L_6(a)$ in (2.12). The introduction of these terms is natural when one is interested in keeping explicitly track of the high operator norms of Φ . \square

Remark 2.5. As already said in the Introduction, if E_a is a Sobolev space, then the weak space E'_a defined in [20] is a strictly larger set, with a strictly weaker norm. We give a simple example in the case where E_a is the Sobolev space of complex-valued periodic functions

$$E_a = H^a(\mathbb{T}^d, \mathbb{C}) := \left\{ u(x) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k e^{ik \cdot x} : \|u\|_a^2 := \sum_{k \in \mathbb{Z}^d} |\hat{u}_k|^2 \langle k \rangle^{2a} < \infty \right\},$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and $\langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}$. Then

$$u(x) := \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-a-\frac{d}{2}} e^{ik \cdot x} \in E'_a \setminus E_a. \quad (2.18)$$

We recall that the weak space E'_a is defined in [20] by means of smoothing operators S_{θ_j} . Moreover it is observed in [20] that different choices of the family S_{θ_j} lead to the same set E'_a with equivalent norms. To prove that u in (2.18) belongs to E'_a , a convenient choice of S_{θ_j} is given by the dyadic Fourier truncations $S_{\theta_j} v(x) := \sum_{|k| \leq 2^j} \hat{v}_k e^{ik \cdot x}$. \square

Remark 2.6. In [19], on page 18 at the end of Section 2.1, regarding his Theorem 2.2.2 Hörmander says: “Most of the papers just quoted state the implicit function theorem not only for Hölder classes but more generally for families of Banach spaces for which the basic calculus lemmas including regularization operators are available, as given in the appendix for Hölder classes. The results proved here can obviously also be stated in that way.”

As already said in the Introduction, Sobolev spaces are not included in the families of Banach spaces that satisfy the properties listed in the Appendix of [19]. In particular, the crucial property in Theorem A.11 on page 44 of [19] does not hold for Sobolev spaces. Given a fixed convex compact set $B \subseteq \mathbb{R}^n$ with nonempty interior, Theorem A.11 of [19] says this:

Theorem A.11 of [19]. *Let u_θ for $\theta > \theta_0$ be a C^∞ function in B and assume that $\|u_\theta\|_{a_i} \leq M\theta^{b_i-1}$, $i = 0, 1$, where $b_0 < 0 < b_1$ and $a_0 < a_1$. Define λ by $\lambda b_0 + (1-\lambda)b_1 = 0$ and set $a = \lambda a_0 + (1-\lambda)a_1$, that is, $a = (a_0 b_1 - a_1 b_0)/(b_1 - b_0)$. If a is not an integer, it follows then that $u = \int_{\theta_0}^\infty u_\theta d\theta$ is in H^a and that $\|u\|_a \leq C_a M$.*

It is not difficult to construct a counterexample to Theorem A.11 of [19] in Sobolev class. For example, on $H^s(\mathbb{R}^d, \mathbb{C})$ take $u_\theta(x) = \int_{\mathbb{R}^d} \varphi(|\xi|/\theta) e^{i\xi \cdot x} d\xi \theta^{-\beta}$ where $\varphi \in C^\infty(\mathbb{R})$, $\text{supp}(\varphi) \subseteq [\frac{1}{2}, \frac{3}{2}]$, with $0 \leq \varphi \leq 1$, and $\varphi(1) = 1$. Let $\beta > \frac{d}{2} + 1$, $\theta_0 = 1$, and fix a_0, a_1 such that $0 \leq a_0 < \beta - \frac{d}{2} - 1 < a_1$. Let $b_i := a_i - \beta + \frac{d}{2} + 1$, $i = 0, 1$, so that $b_0 < 0 < b_1$. Hence u_θ satisfies the assumption $\|u_\theta\|_{a_i} \leq M\theta^{b_i-1}$ of Theorem A.11 of [19]. The theorem gives then $a = \beta - \frac{d}{2} - 1$. However, the function $u = \int_1^\infty u_\theta d\theta$ has Fourier transform $\hat{u}(\xi) = \int_1^\infty \varphi(|\xi|/\theta) \theta^{-\beta} d\theta$. Now $|\hat{u}(\xi)| \geq C|\xi|^{1-\beta}$ for all $|\xi| \geq 1$, and therefore $|\hat{u}(\xi)||\xi|^a \geq C|\xi|^{-\frac{d}{2}}$, whence $u \notin H^a(\mathbb{R}^d)$.

Similarly, on the Sobolev space $H^s(\mathbb{T}^d, \mathbb{C})$ of periodic functions, we take

$$u_\theta(x) = \sum_{k \in \mathbb{Z}^d, \frac{1}{2}\theta \leq |k| \leq \frac{3}{2}\theta} e^{ik \cdot x} \theta^{-\beta}.$$

Let $\beta, a_0, a_1, b_0, b_1, a$ as above. Hence u_θ satisfies the assumption $\|u_\theta\|_{a_i} \leq M\theta^{b_i-1}$ of Theorem A.11 of [19]. The function $u = \int_1^\infty u_\theta d\theta$ has Fourier coefficients $\hat{u}_k = \int_{\frac{2}{3}|k|}^{2|k|} \theta^{-\beta} d\theta \geq C|k|^{1-\beta}$. Therefore $|\hat{u}_k||k|^a \geq C|k|^{-\frac{d}{2}}$, whence $u \notin H^a(\mathbb{T}^d)$.

A consequence of these counterexamples is that Theorem 2.2.2 of [19] does not apply to Sobolev spaces. \square

Remark 2.7. We make an attempt to discuss the consequences of the “velocity” of the sequence (θ_j) of smoothing operators in different Nash-Moser theorems.

In Moser [28], Zehnder [32], and recent improvements like [12, 16], the sequence S_{θ_j} of smoothing operators along the iteration scheme is defined as $\theta_{j+1} = \theta_j^\chi$, with $1 < \chi < 2$ ($\chi = \frac{3}{2}$ in [28]), namely

$$\theta_j = \theta_0^{\chi^j}$$

with $\theta_0 > 1$. Thus θ_j , the ratio θ_{j+1}/θ_j and the difference $\theta_{j+1} - \theta_j$ all diverge to ∞ as $j \rightarrow \infty$.

On the opposite side, in Hörmander [19, 20, 21] the “velocity” of the smoothings is

$$\theta_j = (a + j)^\varepsilon$$

with $a > 0$ large and $\varepsilon \in (0, 1)$ small, so that θ_j diverges, the ratio θ_{j+1}/θ_j tends to 1 and the difference $\theta_{j+1} - \theta_j$ goes to zero. This choice corresponds to a very fine discretization of the continuous real parameter $\theta \in [1, \infty)$ of Nash [30].

An intermediate choice is

$$\theta_j = c^j$$

for some $c > 1$. In this case $\theta_j \rightarrow \infty$, the ratio θ_{j+1}/θ_j is constant and equal to c , and the difference $\theta_{j+1} - \theta_j \rightarrow \infty$. This is the choice in [23] with $c = 2^\varepsilon$ (equations (4.4), (S1), (S2) in [23]). For $c = 2$, it corresponds to the dyadic Littlewood-Paley decomposition, and it is our choice in Theorem 2.1.

The velocity of the sequence θ_j has the following consequences.

If the ratio θ_{j+1}/θ_j diverges to ∞ , then a further loss of regularity is *introduced* in the process of constructing the solution. The main reason of this artificial loss is that the high and low norms of the difference $(S_{\theta_{j+1}} - S_{\theta_j})u$ cannot be sharply estimated in terms of the corresponding powers of θ_j only, but, instead, one has

$$\frac{1}{\theta_{j+1} - \theta_j} \|(S_{\theta_{j+1}} - S_{\theta_j})u\|_b \leq C_{a,b} \max\{\theta_j^{b-a-1}, \theta_{j+1}^{b-a-1}\} \|u\|_a, \quad (2.19)$$

and the maximum is θ_j^{b-a-1} or θ_{j+1}^{b-a-1} according to the (high or low) norm one is estimating. Along the iteration scheme one has to estimate both high and low norms, and the discrepancy between θ_{j+1}^{b-a-1} and θ_j^{b-a-1} generates a loss of regularity. In the particular case $\theta_{j+1} = \theta_j^\chi$, for $b > a+1$ one can write (2.19) in terms of an explicit loss σ of regularity, namely

$$\frac{1}{\theta_{j+1} - \theta_j} \|(S_{\theta_{j+1}} - S_{\theta_j})u\|_b \leq C_{a,b} \theta_j^{b-a-1+\sigma} \|u\|_a \quad (2.20)$$

where $(\chi - 1)(b - a - 1) \leq \sigma$.

Instead, when the ratio θ_{j+1}/θ_j is bounded, (2.19) reduces to

$$\frac{1}{\theta_{j+1} - \theta_j} \|(S_{\theta_{j+1}} - S_{\theta_j})u\|_b \leq C_{a,b} \theta_j^{b-a-1} \|u\|_a. \quad (2.21)$$

If the difference $\theta_{j+1} - \theta_j$ tends to zero, then this can be used to simplify the proof of the convergence of the quadratic error in the telescoping Hörmander scheme. This is

what is done in [20] to obtain bound (15), where $\theta_{j+1} - \theta_j = O(\theta_j^{-N})$ and N is chosen large enough. See also the estimate of the term e_k'' on page 150 in Alinhac-Gérard [3].

In Sobolev class, the orthogonality property (2.9) is somehow related to the velocity of θ_j in the following sense. Consider $E_a = H^a(\mathbb{T}^d)$ or $H^a(\mathbb{R}^d)$. If S_θ is the “crude” Fourier truncation operator

$$S_\theta u(x) = \sum_{k \in \mathbb{Z}^d, |k| \leq \theta} \hat{u}_k e^{ik \cdot x} \quad \text{or} \quad S_\theta u(x) = \int_{|\xi| \leq \theta} \hat{u}(\xi) e^{i\xi \cdot x} d\xi,$$

and $R_0 := S_{\theta_1}$, $R_j := (S_{\theta_{j+1}} - S_{\theta_j})$, then (2.9) holds no matter what the choice of the sequence θ_j is (with $\theta_0 < \theta_1 < \theta_2 < \dots \rightarrow \infty$).

If, instead, S_θ is a smooth Fourier cut-off operator

$$S_\theta u(x) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k \psi\left(\frac{|k|}{\theta}\right) e^{ik \cdot x} \quad \text{or} \quad S_\theta u(x) = \int_{\mathbb{R}^d} \hat{u}(\xi) \psi\left(\frac{|\xi|}{\theta}\right) e^{i\xi \cdot x} d\xi,$$

where $\psi \in C^\infty$, $0 \leq \psi \leq 1$, $\psi = 1$ on $[0, 1]$ and $\psi = 0$ on $[2, \infty)$, then the orthogonality condition (2.9) holds if $\theta_{j+1}/\theta_j \geq c > 1$, and it does not hold if $\theta_{j+1}/\theta_j \rightarrow 1$. These smooth Fourier cut-offs, commonly used in Fourier analysis, are a natural choice when property (iv) of [20] has to be satisfied (in Theorem 2.1, property (iv) of [20] has been replaced by the less demanding inequality (2.5)). \square

3 Proof of Theorem 2.1

Fix $\gamma > 0$ such that $2a_1 + \beta + \gamma \leq 2\alpha$. In this proof we denote by C any constant (possibly different from line to line) depending only on $a_1, a_2, \alpha, \beta, \mu, a_0, \gamma$, which are fixed parameters. Denote, in short,

$$g_j := R_j g \quad \forall j \geq 0. \quad (3.1)$$

By (2.8),

$$\|g_j\|_b \leq C_b 2^{j(b-\beta)} \|g_j\|_\beta \quad \forall b \in [0, +\infty). \quad (3.2)$$

We claim that if $\|g\|_\beta$ is small enough, then we can define a sequence $u_j \in V \cap E_{a_2+c}$ with $u_0 := 0$ by the recursion formula

$$u_{j+1} := u_j + h_j, \quad v_j := S_j u_j, \quad h_j := \Psi(v_j)(g_j + y_j) \quad \forall j \geq 0, \quad (3.3)$$

where $y_0 := 0$,

$$y_1 := -S_1 e_0, \quad y_j := -S_j e_{j-1} - R_{j-1} \sum_{i=0}^{j-2} e_i \quad \forall j \geq 2, \quad (3.4)$$

and $e_j := e'_j + e''_j$,

$$e'_j := \Phi(u_j + h_j) - \Phi(u_j) - \Phi'(u_j)h_j, \quad e''_j := (\Phi'(u_j) - \Phi'(v_j))h_j. \quad (3.5)$$

We prove that there exist positive constants K_1, \dots, K_4 such that, for all $j \geq 0$,

$$\|h_j\|_a \leq K_1 (\|g\|_\beta 2^{-j\gamma} + \|g_j\|_\beta) 2^{j(a-\alpha)} \quad \forall a \in [a_1, a_2], \quad (3.6)$$

$$\|v_j\|_a \leq K_2 \|g\|_\beta 2^{j(a-\alpha)} \quad \forall a \in [a_1 + \beta, a_2 + \beta], \quad (3.7)$$

$$\|u_j - v_j\|_a \leq K_3 \|g\|_\beta 2^{j(a-\alpha)} \quad \forall a \in [0, a_2], \quad (3.8)$$

$$\|u_j\|_\alpha \leq K_4 \|g\|_\beta. \quad (3.9)$$

For $j = 0$, (3.7), (3.8) and (3.9) are trivially satisfied, and (3.6) follows from (3.2) because $h_0 = \Psi(0)g_0$, provided that $C(L_4(a_2) + L_6(a_2)) \leq K_1$.

Now let $k \geq 0$ and assume that, for all $j = 0, \dots, k$, (3.6), (3.7), (3.8), (3.9) hold. First we prove (3.9) at $j = k + 1$. By (2.8) and (3.6) one has for all $n \leq k$, all $j \geq 0$,

$$\|R_j h_n\|_\alpha \leq C 2^{j(\alpha-a)} \|h_n\|_a \leq CK_1 \xi_n 2^{(j-n)(\alpha-a)} \quad \forall a \in [a_1, a_2], \quad (3.10)$$

where $\xi_n := \|g\|_\beta 2^{-n\gamma} + \|g_n\|_\beta$. Since $u_{k+1} = \sum_{n=0}^k h_n$, using (3.10) with $a = a_1$ if $n > j$ and $a = a_2$ if $n \leq j$, we get

$$\|R_j u_{k+1}\|_\alpha \leq \sum_{n=0}^k \|R_j h_n\|_\alpha \leq CK_1 (\varepsilon'_j + \varepsilon''_j) \quad (3.11)$$

where

$$\varepsilon'_j := \sum_{n=j+1}^k \xi_n 2^{-(n-j)(\alpha-a_1)}, \quad \varepsilon''_j := \sum_{n=0}^{\min\{k,j\}} \xi_n 2^{-(j-n)(a_2-\alpha)} \quad (3.12)$$

and $\varepsilon'_j = 0$ for $j+1 > k$ (empty sum). By Hölder inequality,

$$\begin{aligned} \sum_{j=0}^{\infty} \varepsilon_j'^2 &\leq \sum_{j=0}^{\infty} \left(\sum_{n=j+1}^k \xi_n^2 2^{-(n-j)(\alpha-a_1)} \right) \left(\sum_{n=j+1}^k 2^{-(n-j)(\alpha-a_1)} \right) \\ &\leq C \sum_{j=0}^{\infty} \sum_{n=j+1}^k \xi_n^2 2^{-(n-j)(\alpha-a_1)} = C \sum_{n=1}^k \xi_n^2 \sum_{j=0}^{n-1} 2^{-(n-j)(\alpha-a_1)} \\ &\leq C \sum_{n=1}^k \xi_n^2 \leq C(1+A)^2 \|g\|_\beta^2, \end{aligned} \quad (3.13)$$

where the last inequality follows from (2.13), (2.14) and (3.1). Similarly, one proves that $\sum_{j=0}^{\infty} \varepsilon_j''^2 \leq C(1+A)^2 \|g\|_\beta^2$. Thus by (2.9) and (3.11) we deduce that

$$\|u_{k+1}\|_\alpha \leq CK_1(1+A) \|g\|_\beta, \quad (3.14)$$

which gives (3.9) if $CK_1(1+A) \leq K_4$.

Now we prove (3.8) for $j = k + 1$. By (2.4), (2.2) and (3.14) one has

$$\|u_{k+1} - v_{k+1}\|_0 \leq C 2^{-(k+1)\alpha} \|u_{k+1}\|_\alpha \leq CK_1(1+A) \|g\|_\beta 2^{-(k+1)\alpha}. \quad (3.15)$$

By triangular inequality, (2.2) and (3.6) we get

$$\|u_{k+1} - v_{k+1}\|_{a_2} \leq C \|u_{k+1}\|_{a_2} \leq C \sum_{n=0}^k \|h_n\|_{a_2} \leq CK_1 \|g\|_\beta 2^{(k+1)(a_2-\alpha)}. \quad (3.16)$$

Interpolating between 0 and a_2 by (2.6) gives $\|u_{k+1} - v_{k+1}\|_a \leq CK_1(1+A) \|g\|_\beta 2^{(k+1)(a-\alpha)}$ for all $a \in [0, a_2]$. This gives (3.8) if $CK_1(1+A) \leq K_3$.

To prove (3.7) at $j = k + 1$, we use the assumption $a_1 + \beta > \alpha$, (2.3) and (3.14) and we get

$$\|v_{k+1}\|_a \leq C 2^{(k+1)(a-\alpha)} \|u_{k+1}\|_\alpha \leq CK_1(1+A) \|g\|_\beta 2^{(k+1)(a-\alpha)}$$

for all $a \in [a_1 + \beta, a_2 + \beta]$. This gives (3.7) if $CK_1(1+A) \leq K_2$.

Now we prove (3.6) for $j = k + 1$. We begin with proving that

$$\|y_{k+1}\|_b \leq CK_1(K_1 + K_3)M_{123}(a_2 - \mu)\|g\|_\beta^2 2^{(k+1)(b-\beta-\gamma)} \quad \forall b \in [0, a_2 + \beta - \alpha]. \quad (3.17)$$

Since $u_j, v_j, u_j + h_j$ belong to V for all $j = 0, \dots, k$, we use Taylor formula and (2.11) to deduce that, for $j = 0, \dots, k$ and $a \in [0, a_2 - \mu]$,

$$\begin{aligned} \|e_j\|_a &\leq \|h_j\|_{a+\mu}\|h_j\|_{a_0}\{M_1(a) + M_2(a)\|h_j\|_{a_0}\} + \|h_j\|_{a_0}^2\{M_3(a) + M_2(a)\|u_j\|_{a+\mu}\} \\ &\quad + \|h_j\|_{a_0}\|v_j - u_j\|_{a+\mu}\{M_1(a) + M_2(a)\|v_j - u_j\|_{a_0}\} + \|h_j\|_{a+\mu}\|v_j - u_j\|_{a_0}M_1(a) \\ &\quad + \|h_j\|_{a_0}\|v_j - u_j\|_{a_0}\{M_3(a) + M_2(a)\|v_j\|_{a+\mu}\}. \end{aligned} \quad (3.18)$$

Let $p := \max\{0, \beta - \alpha + \mu\}$. For future convenience, note that $p \leq a_1 + \beta - \alpha$ because $0 < a_1 + \beta - \alpha$ and $\mu + \beta - \alpha \leq a_1 + \beta - \alpha$. By assumption, $\gamma \leq 2\alpha - \beta - 2a_1$ and $2\alpha - a_1 < a_2$. Hence

$$\alpha + p + \gamma \leq 3\alpha + p - \beta - 2a_1 \leq 3\alpha + (a_1 + \beta - \alpha) - \beta - 2a_1 = 2\alpha - a_1 < a_2. \quad (3.19)$$

Let $q := a_2 + \beta - \alpha + \mu - p$ (so that $q = a_2$ if $\beta - \alpha + \mu \geq 0$, and $q < a_2$ if $\beta - \alpha + \mu < 0$). For $j = 1, \dots, k$, by (3.6) we have

$$\|u_j\|_q \leq \|u_j\|_{a_2} \leq \sum_{i=0}^{j-1} \|h_i\|_{a_2} \leq K_1\|g\|_\beta \sum_{i=0}^{j-1} 2^{i(a_2-\alpha)} \leq CK_1\|g\|_\beta 2^{j(a_2-\alpha)}, \quad (3.20)$$

while for $j = 0$ we have $u_0 = 0$ by assumption. We consider (3.18) with $a = q - \mu$ (note that $q - \mu \in [0, a_2 - \mu]$). Since $a_0 \leq a_1$, using (3.20), (3.6), (3.8) we have

$$\begin{aligned} \|e_j\|_{a_2+\beta-\alpha-p} &\leq CK_1(K_1 + K_3)\|g\|_\beta^2 \left\{ M_1(a_2 - \mu)2^{j(a_1+q-2\alpha)} \right. \\ &\quad \left. + M_2(a_2 - \mu)2^{j(a_2+2a_1-3\alpha)} + M_3(a_2 - \mu)2^{j(2a_1-2\alpha)} \right\} \end{aligned}$$

provided that $K_1\|g\|_\beta \leq 1$. We assume that $K_1\|g\|_\beta \leq 1$. By the definition of q , the exponents $(a_1 + q - 2\alpha)$, $(a_2 + 2a_1 - 3\alpha)$ and $(2a_1 - 2\alpha)$ are $\leq (a_2 - \alpha - p - \gamma)$ because, by assumption, $2a_1 + \beta + \gamma \leq 2\alpha$. Thus

$$\|e_j\|_{a_2+\beta-\alpha-p} \leq CK_1(K_1 + K_3)M_{123}(a_2 - \mu)\|g\|_\beta^2 2^{j(a_2-\alpha-p-\gamma)}. \quad (3.21)$$

Now we estimate $\|S_{k+1}e_k\|_0$. By (3.9), $\|u_k\|_\mu \leq \|u_k\|_\alpha \leq K_4\|g\|_\beta$, and we assume that $K_4\|g\|_\beta \leq 1$. Since $a_0, \mu \leq a_1$, by (2.2), (3.6), (3.8) and (3.18), using the bound $2a_1 + \beta + \gamma \leq 2\alpha$, we get

$$\|S_{k+1}e_k\|_0 \leq CK_1(K_1 + K_3)M_{123}(0)\|g\|_\beta^2 2^{-(k+1)(\beta+\gamma)}. \quad (3.22)$$

By (2.3) and (3.22) we deduce that

$$\|S_{k+1}e_k\|_b \leq CK_1(K_1 + K_3)M_{123}(0)\|g\|_\beta^2 2^{(k+1)(b-\beta-\gamma)} \quad (3.23)$$

for all $b \in [0, a_2 + \beta - \alpha]$. Now we estimate the other terms in y_{k+1} (see (3.4)). For all $b \in [0, a_2 + \beta - \alpha]$, by (2.8) and (3.21) we have

$$\begin{aligned} \sum_{i=0}^{k-1} \|R_k e_i\|_b &\leq \sum_{i=0}^{k-1} C 2^{k(b-a_2-\beta+\alpha+p)} \|e_i\|_{a_2+\beta-\alpha-p} \\ &\leq CK_1(K_1 + K_3)M_{123}(a_2 - \mu)\|g\|_\beta^2 2^{k(b-a_2-\beta+\alpha+p)} \sum_{i=0}^{k-1} 2^{i(a_2-\alpha-p-\gamma)} \\ &\leq CK_1(K_1 + K_3)M_{123}(a_2 - \mu)\|g\|_\beta^2 2^{k(b-\beta-\gamma)} \end{aligned} \quad (3.24)$$

because $a_2 - \alpha - p - \gamma > 0$ (see (3.19)). The sum of (3.23) and (3.24) completes the proof of (3.17).

Now we are ready to prove (3.6) at $j = k + 1$. By (2.2) and (3.14) we have $\|v_{k+1}\|_{a_1} \leq C\|u_{k+1}\|_{a_1} \leq CK_1\|g\|_\beta$, and we assume that $CK_1\|g\|_\beta \leq \delta_1$, so that $\Psi(v_{k+1})$ is defined. By (3.3), (2.12), (3.2), (3.17), (3.7) one has, for all $a \in [a_1, a_2]$,

$$\begin{aligned} \|h_{k+1}\|_a &\leq C\{K_1(K_1 + K_3)M_{123}(a_2 - \mu)\|g\|_\beta^2 2^{-(k+1)\gamma} + \|g_{k+1}\|_\beta\} \\ &\quad \cdot \{[L_4(a) + L_5(a)]2^{(k+1)(a-\alpha)} + L_6(a)2^{-(k+1)\beta}\} \end{aligned} \quad (3.25)$$

if $K_2\|g\|_\beta \leq 1$. We assume that $K_2\|g\|_\beta \leq 1$. Since $-\beta < a_1 - \alpha$, bound (3.25) implies (3.6) if

$$CL_{456}(a_2) \leq K_1, \quad CL_{456}(a_2)(K_1 + K_3)M_{123}(a_2 - \mu)\|g\|_\beta \leq 1.$$

The induction proof of (3.6), (3.7), (3.8), (3.9) is complete if $K_1, K_2, K_3, K_4, \|g\|_\beta$ satisfy:

$$\begin{aligned} C_*L_{456}(a_2) &\leq K_1; \quad C_*K_1(1 + A) \leq K_i \quad \text{for } i = 2, 3, 4; \quad K_m\|g\|_\beta \leq 1 \quad \text{for } m = 1, 2, 4; \\ C_*K_1\|g\|_\beta &\leq \delta_1; \quad C_*M_{123}(a_2 - \mu)L_{456}(a_2)(K_1 + K_3)\|g\|_\beta \leq 1 \end{aligned} \quad (3.26)$$

where C_* is the largest of the constants appearing above. First we fix $K_1 = C_*L_{456}(a_2)$. Then we fix $K_2 = K_3 = K_4 = C_*K_1(1 + A)$, and finally we fix $\delta > 0$ such that the last five inequalities hold for all $\|g\|_\beta \leq \delta$, namely we fix $\delta = 1/\max\{K_1, K_2, C_*K_1/\delta_1, C_*M_{123}(a_2 - \mu)L_{456}(a_2)(K_1 + K_3)\}$. This completes the proof of (3.6), (3.7), (3.8), (3.9).

The same argument used in (3.10), (3.11), (3.12), (3.13) proves that (u_n) is a Cauchy sequence in E_α . Hence u_n converges to a limit $u \in E_\alpha$, with $\|u\|_\alpha \leq K_4\|g\|_\beta$.

We prove the convergence of the scheme. By (3.4) and (2.7) one proves by induction that

$$\sum_{j=0}^k (e_j + y_j) = e_k + r_k, \quad \text{where } r_k := (I - S_{\theta_k}) \sum_{j=0}^{k-1} e_j, \quad \forall k \geq 1.$$

Hence, by (3.3) and (3.5), recalling that $\Phi'(v_j)\Psi(v_j)$ is the identity map, one has

$$\Phi(u_{k+1}) - \Phi(u_0) = \sum_{j=0}^k [\Phi(u_{j+1}) - \Phi(u_j)] = \sum_{j=0}^k (e_j + g_j + y_j) = G_k + e_k + r_k$$

where $G_k := \sum_{j=0}^k g_j = S_{k+1}g$. By (2.4), (2.2), $\|G_k - g\|_b \rightarrow 0$ as $k \rightarrow \infty$, for all $b \in [0, \beta]$. By (3.18), (3.6), (3.8) and (3.9), $\|e_j\|_{\alpha-\mu} \leq M2^{j(a_1-\alpha)}$ for some $M > 0$, and the series $\sum_{j=0}^\infty \|e_j\|_{\alpha-\mu}$ converges. By (2.4), for all $\rho \in [0, \alpha - \mu)$ we have

$$\|r_k\|_\rho \leq \sum_{j=0}^{k-1} \|(I - S_k)e_j\|_\rho \leq \sum_{j=0}^{k-1} C_\rho 2^{-k(\alpha-\mu-\rho)} \|e_j\|_{\alpha-\mu} \leq C_\rho M 2^{-k(\alpha-\mu-\rho)}, \quad (3.27)$$

so that $\|r_k\|_\rho \rightarrow 0$ as $k \rightarrow \infty$. We have proved that $\|\Phi(u_k) - \Phi(u_0) - g\|_\rho \rightarrow 0$ as $k \rightarrow \infty$ for all ρ in the interval $0 \leq \rho < \min\{\alpha - \mu, \beta\}$. Since $u_k \rightarrow u$ in E_α , it follows that $\Phi(u_k) \rightarrow \Phi(u)$ in $F_{\alpha-\mu}$. This completes the proof of the first part of the theorem.

Now let $c > 0$. Assume that (2.11) holds for all $a \in [0, a_2 + c - \mu]$, and that (2.12) holds for all $a \in [a_1, a_2 + c]$. Assume that $g \in F_{\beta+c}$, with (2.14). By (2.8),

$$\|g_j\|_b \leq C_{b,c} 2^{j(b-\beta-c)} \|g_j\|_{\beta+c} \quad \forall b \geq 0 \quad (3.28)$$

(namely (3.2) holds for $b \in [0, \infty)$, with β replaced by $\beta + c$). Using (2.3), (3.22), (2.8), (3.24), and (3.26), we have

$$\begin{aligned} \|y_{k+1}\|_b &\leq C_b K_1 (K_1 + K_3) M_{123} (a_2 - \mu) \|g\|_\beta^2 2^{(k+1)(b-\beta-\gamma)} \\ &\leq C_b \|g\|_\beta 2^{(k+1)(b-\beta-\gamma)} \quad \forall b \geq 0 \end{aligned} \quad (3.29)$$

(namely (3.17) holds for $b \in [0, \infty)$, with C replaced by C_b , then we use (3.26), recalling that $K_1 = C_* L_{456}(a_2)$). Using (2.3), (3.7) and (3.26), we have

$$\|v_j\|_a \leq C_a K_2 \|g\|_\beta 2^{j(a-\alpha)} \leq C_a 2^{j(a-\alpha)} \quad \forall a \geq a_1 + \beta \quad (3.30)$$

(namely (3.7) holds for $a \in [a_1 + \beta, \infty)$, with K_2 replaced by $C_a K_2$, then use (3.26)). By (3.3), (2.12) (which now holds for $a \in [a_1, a_2 + c]$), (3.28), (3.29), (3.30), and (3.2) for the term containing $L_6(a) \|g_k\|_0$, we deduce that, for all $k \geq 0$,

$$\begin{aligned} \|h_k\|_a &\leq L_{45}(a) (C_{a,c} \|g_k\|_{\beta+c} 2^{k(a-\alpha-c)} + C_a \|g\|_\beta 2^{k(a-\alpha-\gamma)}) + L_6(a) \mathcal{C} 2^{-k\beta} \xi_k \\ &\leq L_{45}(a) C_{a,c} 2^{k(a-\alpha-\lambda)} \eta_k + L_6(a) \mathcal{C} 2^{-k\beta} \psi_k \quad \forall a \in [a_1, a_2 + c], \end{aligned} \quad (3.31)$$

where $L_{45} := L_4 + L_5$, \mathcal{C} is the sum of the two constants C_b at $b = 0$ appearing in (3.2) and (3.29), ξ_k has been defined above as $\xi_k = \|g\|_\beta 2^{-k\gamma} + \|g_k\|_\beta$,

$$\eta_k := \|g_k\|_{\beta+c} + \|g\|_{\beta+c} 2^{-k\gamma/2}, \quad \psi_k := \|g_k\|_\beta + \|g\|_\beta 2^{-k\gamma/2}, \quad \lambda := \frac{c}{N}, \quad (3.32)$$

and N is the smallest positive integer that is $\geq 2c/\gamma$ (so that $\lambda \leq \min\{c, \gamma/2\}$ and $N\lambda = c$). For $a = a_1$, by (3.3), (2.12), (3.28) (which here we use also for the term containing $L_6(a_1) \|g_k\|_0$), (3.29) and (3.30), since $-\beta < a_1 - \alpha$, we obtain

$$\|h_k\|_{a_1} \leq C_c L_{456}(a_1) 2^{k(a_1-\alpha-\lambda)} \eta_k. \quad (3.33)$$

If $N = 1$, then (3.31) gives (3.54) below. If, instead, $N \geq 2$, we repeat the argument and prove recursively for $n = 1, \dots, N$ the following bounds: for all $k \geq 0$, all $a \in [a_1, a_2 + c]$,

$$\|h_k\|_a \leq 2^{k(a-\alpha-n\lambda)} (\mathcal{A}_n(a) \psi_k + \mathcal{B}_n(a) \eta_k) + 2^{-k\beta} L_6(a) \mathcal{C} \psi_k, \quad (3.34)$$

$$\|h_k\|_{a_1} \leq 2^{k(a_1-\alpha-n\lambda)} (\mathcal{E}_n \psi_k + \mathcal{F}_n \eta_k), \quad (3.35)$$

where the coefficients $\mathcal{A}_n(a), \mathcal{B}_n(a), \mathcal{E}_n, \mathcal{F}_n$ are defined recursively, and \mathcal{C} has been defined above as the sum of the two constants C_b at $b = 0$ appearing in (3.2) and (3.29). Estimates (3.31) and (3.33) give (3.34), (3.35) for $n = 1$ with

$$\mathcal{A}_1(a) = \mathcal{E}_1 = 0, \quad \mathcal{B}_1(a) = L_{45}(a) C_{a,c}, \quad \mathcal{F}_1 = L_{456}(a_1) C_c. \quad (3.36)$$

Suppose that (3.34)-(3.35) hold for some $n \in [1, N-1]$. We have to prove that they also hold for $n+1$. By (3.34), since $\psi_k \leq C \|g\|_\beta$, $\eta_k \leq C_c \|g\|_{\beta+c}$, and $a_2 + c - \alpha - n\lambda > 0$,

$$\begin{aligned} \|u_k\|_{a_2+c} &\leq \sum_{j=0}^{k-1} \|h_j\|_{a_2+c} \\ &\leq 2^{k(a_2+c-\alpha-n\lambda)} (\tilde{\mathcal{A}}_n C \|g\|_\beta + \tilde{\mathcal{B}}_n C_c \|g\|_{\beta+c}) + \tilde{L}_6 \mathcal{C} C \|g\|_\beta, \end{aligned} \quad (3.37)$$

where $\tilde{\mathcal{A}}_n := \mathcal{A}_n(a_2 + c)$, $\tilde{\mathcal{B}}_n := \mathcal{B}_n(a_2 + c)$, $\tilde{L}_6 := L_6(a_2 + c)$. By (2.2), $\|v_k\|_{a_2+c} \leq C_c \|u_k\|_{a_2+c}$. Therefore v_k satisfies the same bound (3.37) as u_k , and, by triangle inequality, $\|v_k - u_k\|_{a_2+c}$ also does.

By assumption, (2.11) holds for $a \in [0, a_2 + c - \mu]$. Therefore (3.18) also holds for a in the same interval, and it can be used to estimate $\|e_j\|_{a_2+c-\mu}$. Using (3.6), (3.8), (3.26) for the “low norm” factors $\|h_j\|_{a_1}$, $\|v_j - u_j\|_{a_1}$, and (3.34), (3.37) for the “high norm” factors $\|h_j\|_{a_2+c}$, $\|u_j\|_{a_2+c}$, $\|v_j\|_{a_2+c}$, $\|v_j - u_j\|_{a_2+c}$, we obtain

$$\begin{aligned} \|e_j\|_{a_2+c-\mu} &\leq 2^{j(a_1+a_2-2\alpha+c-n\lambda)} \{ \tilde{\mathcal{A}}_n \tilde{M}_{12} C \|g\|_\beta + \tilde{\mathcal{B}}_n \tilde{M}_{12} C_c \|g\|_{\beta+c} \} \\ &\quad + 2^{j(a_1-\alpha)} \{ \tilde{L}_6 \tilde{\mathcal{C}} \tilde{M}_{12} C \|g\|_\beta + \tilde{M}_3 K_1 \|g\|_\beta \} \end{aligned} \quad (3.38)$$

where $\tilde{M}_i := M_i(a_2 + c - \mu)$, $i = 1, 2, 3$, and $\tilde{M}_{12} := \tilde{M}_1 + \tilde{M}_2$.

By (3.18), (3.6), (3.8), (3.26) we have $\|e_j\|_0 \leq 2^{j(a_1-\alpha)} \|h_j\|_{a_1} M_{123}(0)$. Hence, by (3.35),

$$\|e_j\|_0 \leq 2^{j(2a_1-2\alpha-n\lambda)} \{ \mathcal{E}_n M_{123}(0) \psi_j + \mathcal{F}_n M_{123}(0) \eta_j \}. \quad (3.39)$$

By (2.3), $\|S_{k+1}e_k\|_b \leq C_b 2^{(k+1)b} \|e_k\|_0$ for all $b \geq 0$, and therefore, using (3.39), we obtain an estimate for $\|S_{k+1}e_k\|_b$ for all $b \geq 0$. By (2.8), for all $b \geq 0$,

$$\sum_{j=0}^{k-1} \|R_k e_j\|_b \leq C_{b,c} 2^{k(b-a_2-c+\mu)} \sum_{j=0}^{k-1} \|e_j\|_{a_2+c-\mu},$$

and therefore, using (3.38) and the fact that $(a_1 + a_2 - 2\alpha + c - n\lambda) > 0$, we get an estimate for $\|R_k \sum_{j=0}^{k-1} e_j\|_b$ for all $b \geq 0$. Recalling (3.4), we deduce that, for all $k \geq 0$,

$$\begin{aligned} \|y_{k+1}\|_b &\leq 2^{(k+1)(b-a_2-c+\mu)} \{ \tilde{L}_6 \tilde{\mathcal{C}} \tilde{M}_{12} C_{b,c} \|g\|_\beta + \tilde{M}_3 C_{b,c} K_1 \|g\|_\beta \} \\ &\quad + 2^{(k+1)(b+2a_1-2\alpha-n\lambda)} \{ \mathcal{E}_n M_{123}(0) C_b \psi_k + \mathcal{F}_n M_{123}(0) C_b \eta_k \\ &\quad + \tilde{\mathcal{A}}_n \tilde{M}_{12} C_{b,c} \|g\|_\beta + \tilde{\mathcal{B}}_n \tilde{M}_{12} C_{b,c} \|g\|_{\beta+c} \} \quad \forall b \geq 0. \end{aligned} \quad (3.40)$$

The exponents in (3.40) satisfy $(b-a_2-c+\mu) \leq (b+2a_1-2\alpha-n\lambda)$, because $a_1+a_2-2\alpha > 0$ and $c = N\lambda > n\lambda$. Moreover, $(b+2a_1-2\alpha-n\lambda) \leq (b-\beta-(n+1)\lambda-(\gamma/2))$ because $\lambda \leq \gamma/2$ and $2a_1-2\alpha+\beta+\gamma \leq 0$. Hence, for all $k \geq 0$,

$$\|y_k\|_b \leq 2^{k(b-\beta-(n+1)\lambda-(\gamma/2))} C_{b,c} Y_n \quad \forall b \geq 0, \quad (3.41)$$

where

$$\begin{aligned} Y_n &:= \{ \tilde{\mathcal{A}}_n \tilde{M}_{12} + \tilde{L}_6 \tilde{\mathcal{C}} \tilde{M}_{12} + K_1 \tilde{M}_3 + \mathcal{E}_n M_{123}(0) \} \|g\|_\beta \\ &\quad + \{ \tilde{\mathcal{B}}_n \tilde{M}_{12} + \mathcal{F}_n M_{123}(0) \} \|g\|_{\beta+c}. \end{aligned} \quad (3.42)$$

By (3.3) and (2.12) we estimate $\|h_k\|_a$ for $a \in [a_1, a_2 + c]$. Since $c = N\lambda \geq (n+1)\lambda$, using (3.28), (3.30) for $L_4(a)\|g_k\|_{a+\beta-\alpha} + L_5(a)\|v_k\|_{a+\beta}\|g_k\|_0$, and (3.2) for $L_6(a)\|g_k\|_0$, we get, for all $a \in [a_1, a_2 + c]$,

$$\|\Psi(v_k)g_k\|_a \leq 2^{k(a-\alpha-(n+1)\lambda)} L_{45}(a) C_{a,c} \|g_k\|_{\beta+c} + 2^{-k\beta} L_6(a) C \|g_k\|_\beta. \quad (3.43)$$

Using (3.41), (3.30) for $L_4(a)\|y_k\|_{a+\beta-\alpha} + L_5(a)\|v_k\|_{a+\beta}\|y_k\|_0$ and (3.29) for $L_6(a)\|y_k\|_0$, we get, for all $a \in [a_1, a_2 + c]$,

$$\|\Psi(v_k)y_k\|_a \leq 2^{k(a-\alpha-(n+1)\lambda)} L_{45}(a) C_{a,c} Y_n 2^{-k\gamma/2} + 2^{-k\beta} L_6(a) C \|g\|_\beta 2^{-k\gamma}. \quad (3.44)$$

Recalling that $K_1 = C_* L_{456}(a_2)$ and the definition (3.32) of ψ_k, η_k , the sum of (3.43) and (3.44) gives (3.34) at $n + 1$, with

$$\mathcal{A}_{n+1}(a) = L_{45}(a)C_{a,c}(\tilde{\mathcal{A}}_n \tilde{M}_{12} + \tilde{L}_6 \tilde{M}_{12} + L_{456}(a_2) \tilde{M}_3 + \mathcal{E}_n M_{123}(0)), \quad (3.45)$$

$$\mathcal{B}_{n+1}(a) = L_{45}(a)C_{a,c}(1 + \tilde{\mathcal{B}}_n \tilde{M}_{12} + \mathcal{F}_n M_{123}(0)). \quad (3.46)$$

Using (3.30), (3.28) also for the term $L_6(a_1)\|g_k\|_0$, we get

$$\|\Psi(v_k)g_k\|_{a_1} \leq 2^{k(a_1 - \alpha - (n+1)\lambda)} L_{456}(a_1)C_c \|g_k\|_{\beta+c}. \quad (3.47)$$

Using (3.41), (3.30) also for the term $L_6(a_1)\|y_k\|_0$, we get

$$\|\Psi(v_k)y_k\|_{a_1} \leq 2^{k(a_1 - \alpha - (n+1)\lambda)} L_{456}(a_1)C_c Y_n 2^{-k\gamma/2}. \quad (3.48)$$

The sum of the last two bounds gives (3.35) at $n + 1$, with

$$\mathcal{E}_{n+1} = L_{456}(a_1)C_c(\tilde{\mathcal{A}}_n \tilde{M}_{12} + \tilde{L}_6 \tilde{M}_{12} + L_{456}(a_2) \tilde{M}_3 + \mathcal{E}_n M_{123}(0)) \quad (3.49)$$

$$\mathcal{F}_{n+1} = L_{456}(a_1)C_c(1 + \tilde{\mathcal{B}}_n \tilde{M}_{12} + \mathcal{F}_n M_{123}(0)). \quad (3.50)$$

Let

$$Z := L_{456}(a_1)C_c M_{123}(0) + \tilde{L}_{45} \tilde{C}_c \tilde{M}_{12}, \quad X := \tilde{L}_6 \tilde{M}_{12} + L_{456}(a_2) \tilde{M}_3, \quad (3.51)$$

where the constant C_c in (3.51) is the one of (3.49)-(3.50), and the constant \tilde{C}_c is the constant $C_{a,c}$ of (3.45)-(3.46) evaluated at $a = a_2 + c$. By induction, the recursive system (3.45), (3.46), (3.49), (3.50) with the initial values (3.36) gives

$$\mathcal{A}_n(a) = L_{45}(a)C_{a,c}X \sum_{j=0}^{n-2} Z^j, \quad \mathcal{B}_n(a) = L_{45}(a)C_{a,c} \sum_{j=0}^{n-1} Z^j, \quad (3.52)$$

$$\mathcal{E}_n = L_{456}(a_1)C_c X \sum_{j=0}^{n-2} Z^j, \quad \mathcal{F}_n = L_{456}(a_1)C_c \sum_{j=0}^{n-1} Z^j \quad (3.53)$$

for all $n \geq 2$. The iteration ends at $n = N$, and, since $N\lambda = c$, we obtain for all $k \geq 0$

$$\|h_k\|_a \leq 2^{k(a - \alpha - c)} (\mathcal{A}_N(a)\psi_k + \mathcal{B}_N(a)\eta_k) + 2^{-k\beta} L_6(a)C\psi_k \quad \forall a \in [a_1, a_2 + c]. \quad (3.54)$$

The argument used in (3.10)-(3.13) (now with $a_1 + c, \alpha + c, a_2 + c$ instead of a_1, α, a_2 , and bound (3.54) instead of (3.6)) proves that (u_n) is a Cauchy sequence in $E_{\alpha+c}$, and its limit u satisfies

$$\|u\|_{\alpha+c} \leq C(c) \{(\tilde{L}_6 + \tilde{\mathcal{A}}_N)(1 + A)\|g\|_{\beta} + \tilde{\mathcal{B}}_N(1 + A_c)\|g\|_{\beta+c}\} \quad (3.55)$$

for some constant $C(c)$ depending on c . The proof of Theorem 2.1 is complete.

4 Application to quasi-linear perturbations of KdV

We use Theorem 2.1 to improve the regularity in the results of exact controllability and local well-posedness for the Cauchy problem of quasi-linear perturbations of KdV obtained in [7].

We consider equations of the form

$$u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0 \quad (4.1)$$

where the nonlinearity $\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx})$ is at least quadratic around $u = 0$, namely the real-valued function $\mathcal{N} : \mathbb{T} \times \mathbb{R}^4 \rightarrow \mathbb{R}$ satisfies

$$|\mathcal{N}(x, z_0, z_1, z_2, z_3)| \leq C|z|^2 \quad \forall z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4, \quad |z| \leq 1. \quad (4.2)$$

We assume that the dependence of \mathcal{N} on u_{xx}, u_{xxx} is Hamiltonian, while no structure is required on its dependence on u, u_x . More precisely, we assume that

$$\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = \mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) + \mathcal{N}_0(x, u, u_x) \quad (4.3)$$

where

$$\begin{aligned} \mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) &= \partial_x \{(\partial_u \mathcal{F})(x, u, u_x)\} - \partial_{xx} \{(\partial_{u_x} \mathcal{F})(x, u, u_x)\} \\ &\text{for some function } \mathcal{F} : \mathbb{T} \times \mathbb{R}^2 \rightarrow \mathbb{R}. \end{aligned} \quad (4.4)$$

Note that the case $\mathcal{N} = \mathcal{N}_1$, $\mathcal{N}_0 = 0$ corresponds to the Hamiltonian equation $\partial_t u = \partial_x \nabla H(u)$ where the Hamiltonian is

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} u_x^2 dx + \int_{\mathbb{T}} \mathcal{F}(x, u, u_x) dx \quad (4.5)$$

and ∇ denotes the $L^2(\mathbb{T})$ -gradient. The unperturbed KdV is the case $\mathcal{F} = -\frac{1}{6}u^3$.

Theorem 4.1 (Exact controllability). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be a nonempty open set. There exist positive universal constants r_1, s_1 such that, if \mathcal{N} in (4.1) is of class C^{r_1} in its arguments and satisfies (4.2), (4.3), (4.4), then there exists a positive constant δ_* depending on T, ω, \mathcal{N} with the following property.*

Let $u_{in}, u_{end} \in H^{s_1}(\mathbb{T}, \mathbb{R})$ with

$$\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1} \leq \delta_*.$$

Then there exists a function $f(t, x)$ satisfying

$$f(t, x) = 0 \quad \text{for all } x \notin \omega, \text{ for all } t \in [0, T],$$

belonging to $C([0, T], H_x^{s_1}) \cap C^1([0, T], H_x^{s_1-3}) \cap C^2([0, T], H_x^{s_1-6})$ such that the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = f & \forall (t, x) \in [0, T] \times \mathbb{T} \\ u(0, x) = u_{in}(x) \end{cases} \quad (4.6)$$

has a unique solution $u(t, x)$ belonging to $C([0, T], H_x^{s_1}) \cap C^1([0, T], H_x^{s_1-3}) \cap C^2([0, T], H_x^{s_1-6})$, which satisfies

$$u(T, x) = u_{end}(x), \quad (4.7)$$

and

$$\begin{aligned} \|u, f\|_{C([0, T], H_x^{s_1})} + \|\partial_t u, \partial_t f\|_{C([0, T], H_x^{s_1-3})} + \|\partial_{tt} u, \partial_{tt} f\|_{C([0, T], H_x^{s_1-6})} \\ \leq C_{s_1} (\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1}) \end{aligned} \quad (4.8)$$

for some $C_{s_1} > 0$ depending on $s_1, T, \omega, \mathcal{N}$.

Moreover, the universal constant $\tau_1 := r_1 - s_1 > 0$ has the following property. For all $r \geq r_1$, all $s \in [s_1, r - \tau_1]$, if, in addition to the previous assumptions, \mathcal{N} is of class C^r and $u_{in}, u_{end} \in H_x^s$, then u, f belong to $C([0, T], H_x^s) \cap C^1([0, T], H_x^{s-3}) \cap C^2([0, T], H_x^{s-6})$ and (4.8) holds with s instead of s_1 .

Theorem 4.2 (Local existence and uniqueness). *There exist positive universal constants r_0, s_0 such that, if \mathcal{N} in (4.1) is of class C^{r_0} in its arguments and satisfies (4.2), (4.3), (4.4), then the following property holds. For all $T > 0$ there exists $\delta_* > 0$ such that for all $u_{in} \in H_x^{s_0}$ satisfying*

$$\|u_{in}\|_{s_0} \leq \delta_*, \quad (4.9)$$

the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0, & (t, x) \in [0, T] \times \mathbb{T} \\ u(0, x) = u_{in}(x) \end{cases} \quad (4.10)$$

has one and only one solution $u \in C([0, T], H_x^{s_0}) \cap C^1([0, T], H_x^{s_0-3}) \cap C^2([0, T], H_x^{s_0-6})$. Moreover

$$\|u\|_{C([0, T], H_x^{s_0})} + \|\partial_t u\|_{C([0, T], H_x^{s_0-3})} + \|\partial_{tt} u\|_{C([0, T], H_x^{s_0-6})} \leq C_{s_0} \|u_{in}\|_{s_0} \quad (4.11)$$

for some $C_{s_0} > 0$ depending on s_0, T, \mathcal{N} .

Moreover the universal constant $\tau_0 := r_0 - s_0 > 0$ has the following property. For all $r \geq r_0$, all $s \in [s_0, r - \tau_0]$, if, in addition to the previous assumptions, \mathcal{N} is of class C^r and $u_{in} \in H_x^s$, then u belongs to $C([0, T], H_x^s) \cap C^1([0, T], H_x^{s-3}) \cap C^2([0, T], H_x^{s-6})$ and (4.11) holds with s instead of s_0 .

Proof of Theorem 4.1. Define

$$P(u) := u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}). \quad (4.12)$$

and

$$\Phi(u, f) := \begin{pmatrix} P(u) - \chi_\omega f \\ u(0) \\ u(T) \end{pmatrix} \quad (4.13)$$

so that the problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = f & \forall (t, x) \in [0, T] \times \mathbb{T} \\ u(0, x) = u_{in}(x) \\ u(T, x) = u_{end}(x) \end{cases} \quad (4.14)$$

is written as $\Phi(u, f) = (0, u_{in}, u_{end})$. The linearized operator $\Phi'(u, f)[h, \varphi]$ at the point (u, f) in the direction (h, φ) is

$$\Phi'(u, f)[h, \varphi] := \begin{pmatrix} P'(u)[h] - \chi_\omega \varphi \\ h(0) \\ h(T) \end{pmatrix}. \quad (4.15)$$

We define the scales of Banach spaces

$$E_s := X_s \times X_s, \quad X_s := C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^{s+3}) \cap C^2([0, T], H_x^s) \quad (4.16)$$

and

$$F_s := \{g = (g_1, g_2, g_3) : g_1 \in C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^s), g_2, g_3 \in H_x^{s+6}\} \quad (4.17)$$

equipped with the norms

$$\|u, f\|_{E_s} := \|u\|_{X_s} + \|f\|_{X_s}, \quad \|u\|_{X_s} := \|u\|_{T, s+6} + \|\partial_t u\|_{T, s+3} + \|\partial_{tt} u\|_{T, s} \quad (4.18)$$

and

$$\|g\|_{F_s} := \|g_1\|_{T, s+6} + \|\partial_t g_1\|_{T, s} + \|g_2, g_3\|_{s+6}. \quad (4.19)$$

In Theorem 4.5 of [7], the following right inversion result for the linearized operator in (4.15) is proved.

Proposition 4.3. *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. There exist two universal constants $\tau, \sigma \geq 3$ and a positive constant δ_* depending on T, ω with the following property.*

Let $s \in [0, r - \tau]$, where r is the regularity of the nonlinearity \mathcal{N} . Let $g = (g_1, g_2, g_3) \in F_s$, and let $(u, f) \in E_{s+\sigma}$, with $\|u\|_{X_\sigma} \leq \delta_$. Then there exists $(h, \varphi) := \Psi(u, f)[g] \in E_s$ such that*

$$P'(u)[h] - \chi_\omega \varphi = g_1, \quad h(0) = g_2, \quad h(T) = g_3, \quad (4.20)$$

and

$$\|h, \varphi\|_{E_s} \leq C_s (\|g\|_{F_s} + \|u\|_{X_{s+\sigma}} \|g\|_{F_0}) \quad (4.21)$$

where C_s depends on s, T, ω .

We define the smoothing operators S_j , $j = 0, 1, 2, \dots$ as

$$S_j u(x) := \sum_{|k| \leq 2^j} \widehat{u}_k e^{ikx} \quad \text{where} \quad u(x) = \sum_{k \in \mathbb{Z}} \widehat{u}_k e^{ikx}$$

The definition of S_j extends in the obvious way to functions $u(t, x) = \sum_{k \in \mathbb{Z}} \widehat{u}_k(t) e^{ikx}$ depending on time. Since S_j and ∂_t commute, the smoothing operators S_j are defined on the spaces E_s, F_s defined in (4.16)-(4.17) by setting $S_j(u, f) := (S_j u, S_j f)$ and similarly on $z = (v, \alpha, \beta)$. One easily verifies that S_j satisfies (2.1)-(2.5) and (2.9) on E_s and F_s .

By (4.13), observe that $\Phi(u, f) := (P(u) - \chi_\omega f, u(0), u(T))$ belongs to F_s when $(u, f) \in E_{s+3}$, $s \in [0, r - 6]$, with $\|u\|_{T, 4} \leq 1$. Its second derivative in the directions (h, φ) and (w, ψ) is

$$\Phi''(u, f)[(h, \varphi), (w, \psi)] = \begin{pmatrix} P''(u)[h, w] \\ 0 \\ 0 \end{pmatrix}.$$

For u in a fixed ball $\|u\|_{X_1} \leq \delta_0$, with δ_0 small enough, we estimate

$$\|P''(u)[h, w]\|_{F_s} \lesssim_s (\|h\|_{X_1} \|w\|_{X_{s+3}} + \|h\|_{X_{s+3}} \|w\|_{X_1} + \|u\|_{X_{s+3}} \|h\|_{X_1} \|w\|_{X_1}) \quad (4.22)$$

for all $s \in [0, r - 6]$. We fix $V = \{(u, f) \in E_3 : \|(u, f)\|_{E_3} \leq \delta_0\}$, $\delta_1 = \delta_*$,

$$a_0 = 1, \quad \mu = 3, \quad a_1 = \sigma, \quad \alpha = \beta > 2\sigma, \quad a_2 > 2\alpha - a_1 \quad (4.23)$$

where δ_*, σ, τ are given by Proposition 4.3, and $r \geq r_1 := a_2 + \tau$ is the regularity of \mathcal{N} . The right inverse Ψ in Proposition 4.3 satisfies the assumptions of Theorem 2.1. Let

$u_{in}, u_{end} \in H_x^{\beta+6}$, with $\|u_{in}, u_{end}\|_{H_x^{\beta+6}}$ small enough. Let $g := (0, u_{in}, u_{end})$, so that $g \in F_\beta$ and $\|g\|_{F_\beta} \leq \delta$. Since g does not depend on time, it satisfies (2.13).

Thus by Theorem 2.1 there exists a solution $(u, f) \in E_\alpha$ of the equation $\Phi(u, f) = g$, with $\|u, f\|_{E_\alpha} \leq C\|g\|_{F_\beta}$ (and recall that $\beta = \alpha$). We fix $s_1 := \alpha + 6$, and (4.8) is proved.

We have found a solution (u, f) of the control problem (4.14). Now we prove that u is the unique solution of the Cauchy problem (4.6), with that given f . Let u, v be two solutions of (4.6) in E_{s_1-6} . We calculate

$$P(u) - P(v) = \int_0^1 P'(v + \lambda(u - v)) d\lambda [u - v] =: \mathcal{L}(u, v)[u - v].$$

The linear operator $\mathcal{L}(u, v)$ has the same structure as the operator \mathcal{L}_0 in (2.12) of [7]. Since u and v both satisfy the Cauchy problem (4.6), we have $\mathcal{L}(u, v)[u - v] = 0$ and $(u - v)(0) = 0$. Hence the well-posedness result in Lemma 6.7 of [7] implies $(u - v)(t) = 0$ for all $t \in [0, T]$. This completes the proof of Theorem 4.1. \square

Proof of Theorem 4.2. We define

$$E_s := C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^{s+3}) \cap C^2([0, T], H_x^s), \quad (4.24)$$

$$F_s := \{(g_1, g_2) : g_1 \in C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^s), g_2 \in H_x^{s+6}\} \quad (4.25)$$

equipped with norms

$$\|u\|_{E_s} := \|u\|_{T, s+6} + \|\partial_t u\|_{T, s+3} + \|\partial_{tt} u\|_{T, s} \quad (4.26)$$

$$\|(g_1, g_2)\|_{F_s} := \|g_1\|_{T, s+6} + \|\partial_t g_1\|_{T, s} + \|\alpha\|_{s+6}, \quad (4.27)$$

and $\Phi(u) := (P(u), u(0))$, where P is defined in (4.12). Given $g := (0, u_{in}) \in F_{s_0}$, the Cauchy problem (4.10) writes $\Phi(u) = g$. We fix $V := \{u \in E_3 : \|u\|_{E_3} \leq \delta_0\}$, where δ_0 is the same as in the proof of Theorem 4.1; we fix $a_0, \mu, a_1, \alpha, \beta, a_2$ like in (4.23), where σ is now the constant appearing in Lemma 6.7 of [7], $\tau = \sigma + 9$ by Lemmas 2.1 and 6.7 of [7] (combined with the definition of the spaces E_s, F_s), $r \geq r_0 := a_2 + \tau$ is the regularity of \mathcal{N} , and δ_1 is small enough to satisfy the assumption $\delta(0) \leq \delta_*$ in Lemma 6.7 of [7].

Assumption (2.12) about the right inverse of the linearized operator is satisfied by Lemmas 6.7 and 2.1 of [7]. We fix $s_0 := \alpha + 6$. Then Theorem 2.1 applies, giving the existence part of Theorem 4.2. The uniqueness of the solution is proved exactly as in the proof of Theorem 4.1. This completes the proof of Theorem 4.2. \square

Remark 4.4. The approach to control and Cauchy problems that we have used in the proof of Theorems 4.1 and 4.2 also applies to other equations.

With Montalto we prove in [8] a similar result for Hamiltonian, quasi-linear perturbations of the Schrödinger equation on the torus in dimension one, using Theorem 2.1.

Theorem 2.1 could also be used as an alternative approach, based on a different non-linear scheme, to prove the controllability result for gravity capillary water waves in [2].

Moreover Theorem 2.1 can be used to solve quasi-periodic versions of nonlinear ODEs like the equation of characteristics for quasi-periodic transport equations, namely equations of the form $\omega \cdot \partial_\varphi u(\varphi, x) = V(\varphi, x + u(\varphi, x))$ on the torus $(\varphi, x) \in \mathbb{T}^{n+1}$, where $\omega \in \mathbb{R}^n$ is a Diophantine vector (see [6]). \square

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